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Technical Report No. 89-49

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PENNSTATE

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Technical Report No. 89-49

November 1989

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Research sponsored by the U.S. Army Research Office under Grant DAAL03-89-K-0139.

SPECTRAL ANALYTIC METHODS FOR THE ESTIMATION OF NUMBER OF SIGNALS AND DIRECTIONS OF ARRIVAL

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ABSTRACT

Consider the model

$$\underset{\sim}{x}(t) = A\underset{\sim}{s}(t) + \underset{\sim}{n}(t), \quad t = 1,\ldots,N$$

where at time t, $\underset{\sim}{x}(t)$ is a p-complex vector of observations, $\underset{\sim}{s}(t)$ is a q-complex vector of unobservable signals emitted from q sources and $\underset{\sim}{n}(t)$ is a noise p-complex vector variable. It is assumed that $\{\underset{\sim}{s}(t)\}$ and $\{\underset{\sim}{n}(t)\}$ are independent sequences with

$$E[\underset{\sim}{n}(t)\underset{\sim}{n}(t)^*] = \sigma^2 I, \quad E[\underset{\sim}{s}(t)\underset{\sim}{s}(t^*)] = \Gamma$$
$$\Rightarrow E[\underset{\sim}{x}(t)\underset{\sim}{x}(t)^*] = A\Gamma A^* + \sigma^2 I$$

The matrix A has a special structure with its k-th column $\underset{\sim}{a}_k$ of the form

$$\underset{\sim}{a}_k' = (1, e^{-j\omega_0\tau_k}, \ldots, e^{-j\omega_0(p-1)\tau_k})$$

where $j = \sqrt{-1}$, $\tau_k = c^{-1}\Delta\sin\theta_k$, θ_k being the direction of arrival of the signal from the k-th source. The problem is the estimation of q (the number of sources) and θ_i's (the directions of arrival).

The paper presents an information theoretic criterion to decide on q, the number of sources, and a spectral analytic method to estimate the τ_i's. The proposed method is straight forward and has some advantages over the search methods proposed in the literature for the latter problem. For instance, our method works even when the signals are coherent, i.e., the covariance matrix Γ of signal is singular, whereas the MUSIC algorithm generally used in searching for the estimates of τ_k is not applicable.

Key Words and Phrases: Akaike information criterion, Direction of arrival, General information criterion, Signal processing.

AMS Classification Index: Primary 62F12: Secondary 62H12

1. INTRODUCTION

The problems of estimation of the number of signals and the directions of their arrival have been considered by a number of authors under the model

$$\underset{\sim}{x}(t) = A\underset{\sim}{s}(t) + \underset{\sim}{n}(t), \quad t = 1,\ldots,N \tag{1.1}$$

where at time t,

$\underset{\sim}{x}(t)$ is a p-complex vector of observations received by p-sensors uniformly spaced,

$\underset{\sim}{s}(t)$ is a q-complex vector of unobservable signals, and

$\underset{\sim}{n}(t)$ is an additive p-complex noise vector.

It is assumed that $\{\underset{\sim}{n}(t)\}$, $\{\underset{\sim}{s}(t)\}$, $t = 1,\ldots,N$, are independent sequences of iid random variables such that

$$E[\underset{\sim}{n}(t)] = 0, \quad E[\underset{\sim}{n}(t)\underset{\sim}{n}(t)^*] = \sigma^2 I$$
$$E[\underset{\sim}{s}(t)] = 0, \quad E[s(t)s(t)^*] = \Gamma \tag{1.2}$$

where the star sign indicates the complex conjugate. Then we have

$$\Sigma = E[\underset{\sim}{x}(t)\underset{\sim}{x}(t)^*] = A\Gamma A^* + \sigma^2 I. \tag{1.3}$$

We generally assume that the signals are not coherent, i.e., the matrix Γ is positive definite. However, this condition is not necessary for the new method of estimation we develop in Section 4 of this paper.

The p×q matrix A has the special structure

$$A = (\underset{\sim}{a}_1 : \ldots : a_q)$$
$$\underset{\sim}{a}_q = \underset{\sim}{a}(\tau_k) = (1, e^{-j\omega_0 \tau_k}, \ldots, e^{-j\omega_0 (p-1)\tau_k}) \tag{1.4}$$

where $j = \sqrt{-1}$, $\tau_k = C^{-1}\Delta \sin\theta_k$, c = speed of propogation, θ_k is the direction of arrival (DOA) from the k-th source and Δ is the inter sensor distance. We may take ω_0 to be unity without loss of generality.

There are two important problems connected with the model (1.1, 1.2). One is the estimation of q, the number of signals (i.e., the cho·ce of the model), and another is the estimation of $\tau_1,\ldots,\tau_q$ providing the estimates of $\theta_1, \ldots, \theta_q$, the directions of arrival. We discuss these two problems with references to earlier work and propose new solutions.

2. INFORMATION THEORETIC CRITERIA FOR MODEL SELECTION

In view of the assumptions (1.2), if we assume that $\underset{\sim}{n}(t)$ has a p-variate complex normal distribution, then the statistic

$$S = N^{-1} \Sigma \, \underset{\sim}{x}(t) x(t)^* \tag{2.1}$$

has a complex Wishart distribution with N degrees of freedom and covariance matrix $\Sigma = A\Gamma A^* + \sigma^2 I$ as in (1.3). If the covariance matrix of the signals is nonsingular, then the number of signals is $q = p-s$ where s is the multiplicity of the smallest root of Σ. Hence the problem of estimating q can be studied within the framework of testing the equality of a given number of smallest eigen values of Σ, for which a statisfactory solution exists (see for instance Anderson (1963), Ligget (1973), Rao (1973, 1983), Schmidt (1981), Tuft and Kumaresan (1980), Wax, Shan and Kailath (1984)).

The log likelihood based on observed data, apart from an additive constant, is

$$\ell_N^{(q)} = -\frac{N}{2}[\log |\Sigma| + \operatorname{tr} \Sigma^{-1} S] \tag{2.2}$$

where q indicates the number of signals. The maximum of (2.3) for given q is

$$\hat{\ell}_N^{(q)} = -\frac{N}{2}\left[\sum_1^q \log \hat{\lambda}_i + (p-q)\log \frac{\hat{\lambda}_{q+1} + \ldots + \hat{\lambda}_p}{p-q}\right] \tag{2.3}$$

where $\hat{\lambda}_1 > \ldots > \hat{\lambda}_p$ are the eigen values of S assumed to be distinct (with probability 1). The likelihood ratio statistic for testing the equality of the last p-q eigen values of Σ is

$$G_N^{(q)} = 2(\ell_N^{(p)} - \ell_N^{(q)}) = N[(p-q)\log \frac{\hat{\lambda}_{q+1}+\ldots+\hat{\lambda}_p}{p-q} - \log(\hat{\lambda}_{q+1} \ldots \hat{\lambda}_p)] \tag{2.4}$$

which is asymptotically distributed as χ^2 on $(p-q)^2 - 1$ degrees of freedom (d.f.). Let

$$\hat{q} = \max \{q: G_N^{(q)} \leq C_\alpha\}$$

where C_α is the upper tail α% point of χ^2 on $(p-q)^2 - 1$ d.f. Then an upper $(1-\alpha)$% confidence limit to q is $\hat{q}$.

However, to get a point estimate of q, we can use a general information criterion (GIC) developed by Zhao, Krishnaiah and Bai (1986)

$$GIC(k) = 2\ell_N^{(k)} - \nu(k)C_N \tag{2.5}$$

where $\nu(k)$ is the number of free parameters in the model and C_N is a function of N such that as $N \to \infty$

$$\frac{C_N}{N} \longrightarrow 0 \quad \text{and} \quad \frac{C_N}{\log\log N} \longrightarrow \infty. \tag{2.6}$$

In the present problem $\nu(k) = q(2p-q) + 1$, and the decision (estimation) rule is the choose $q = \hat{q}$ where

$$GIC(\hat{q}) = \max_k GIC(k). \tag{2.7}$$

It is shown by Zhao, Krishnaiah and Bai (1986) that under the condition (2.6), the GIC rule (2.7) is strongly consistent, i.e.,

$$\hat{q} \to q \quad \text{a.s. as} \quad n \to \infty. \tag{2.8}$$

Criteria of the kind (2.7) for model selection have been used earlier with special choices:

$$\begin{aligned} C_N &= 2, \quad \text{Akaike} \\ &= \log N, \quad \text{Kashyap, Schwartz, Rissanen} \\ &= c \operatorname{loglog} n, \text{ for some } c > 2, \quad \text{Hannan and Quinn.} \end{aligned}$$

All these choices except Akaike's give a consistent estimate of q. There are, however, no adequate studies on the choice of C_N in finite samples. Rao and Wu (1989) indicated an adaptive method for choosing C_N in a similar context. Further research in this direction is necessary.

3. ESTIMATION OF DOA: NOISE SPACE METHOD

For a given q, the eigen values of Σ are of the form

$$\lambda_1 \geq \dots \geq \lambda_q > \lambda_{q+1} = \dots = \lambda_p (= \sigma^2). \tag{3.1}$$

Let $\underset{\sim}{e}_1, \dots, \underset{\sim}{e}_q, \underset{\sim}{e}_{q+1}, \dots \underset{\sim}{e}_p$ be the corresponding eigen vectors and define the matrices

$$E_s = (\underset{\sim}{e}_1, \dots, \underset{\sim}{e}_q), \quad E_n = (\underset{\sim}{e}_{q+1}, \dots, \underset{\sim}{e}_n). \tag{3.2}$$

The spaces spanned by the vectors in E_s and E_n may be called the signal and noise eigen spaces, respectively. It is easily seen that

$$\underset{\sim}{a}(\tau)'E_n = 0 \iff D(\tau) \triangleq \underset{\sim}{a}(\tau)'E_n E_n' \underset{\sim}{a}(\tau) = 0 \tag{3.3}$$

for $\tau = \tau_i$, $i = 1,\ldots,q$, as defined in (1.4). In practice we have only the estimate S (as defined in (2.1)) of Σ. The eigen values and eigen vectors of S

$$\begin{gathered} \hat{\lambda}_1 > \ldots > \hat{\lambda}_q > \ldots > \hat{\lambda}_p \\ \hat{e}_1, \ldots, \hat{e}_q, \ldots, \hat{e}_q \end{gathered} \tag{3.4}$$

provide consistent estimates of the corresponding eigen values (3.1) and eigen vectors (3.2) of the true Σ. An estimate of E_n spanning the noise eigen space is

$$\hat{E}_n = (\hat{\underset{\sim}{e}}_{q+1},\ldots,\hat{\underset{\sim}{e}}_p). \tag{3.5}$$

The function

$$\hat{D}(\tau) = a(\tau)'\hat{E}_n\hat{E}_n a(\tau) \tag{3.6}$$

which is an estimate of $D(\tau)$ defined in (3.3) may not vanish for any τ, but is likely to be small in the neighborhood of $\tau_1,\ldots,\tau_q$. Then they may be estimated by plotting $1/\hat{D}(\tau)$ and locating the peaks, which is the basis of the MUSIC algorithm (see papers by Bienvenu (1979), Schmidt (1981) and Wax, Shan and Kailath (1984)). As the solutions are obtained essentially by a search method, their analytical characterization is complicated so that the statistical properties of the estimates are difficult to study. We propose an alternative simple computational technique which provides definitive estimates and which enable us to study their asymptotic properties.

We note that there exists a matrix G of order $p \times r$ (with r written for $p-q$)

$$G = (\underset{\sim}{G}_1, \ldots, \underset{\sim}{G}_r) \tag{3.7}$$

where $\underset{\sim}{G}_k$ is of the form

$$\underset{\sim}{G}_k = (0, \ldots, 0, g_1, \ldots, g_{q+1}, 0, \ldots, 0)', \quad k = 1, \ldots, r, \tag{3.8}$$

with the first k and the last (p-k-q-1) components as zeroes

$g_{q+1} > 0$, $g_1\bar{g}_1 + \ldots + g_{q+1}\bar{g}_{q+1} = 1$ such that $\underset{\sim}{a}(\tau)'G = 0$, for $\tau = \tau_1, \ldots, \tau_k$, and $\exp(j\tau_1), \ldots, \exp(j\tau_k)$ are the solutions of the polynomial equation

$$g_{q+1}z^q + \ldots + g_1 = 0. \tag{3.9}$$

Then the strategy is to estimate $\underset{\sim}{g} = (g_1, \ldots, g_{q+1})$ and use them in (3.9) to obtain the roots which provide the estimates of $\tau_1, \ldots, \tau_k$.

Since the columns of G generate (i.e., provide a basis of) the noise eigen space E_n of Σ, the statistical problem of estimating G may be thought of as fitting a basis of the type $G_1, \ldots, G_r$ to the estimated noise eigen space $\hat{E}_n$ given in (3.5). The mathematical problem may be formulated as that of minimizing the Euclidean norm

$$||\hat{E}_n - GB|| \tag{3.10}$$

with respect to a matrix G of the type (3.7) and an arbitrary matrix B of order $r \times r$.

As the optimization problem (3.10) is complicated, we suggest some alternative procedures. By Householder transformation (for which reliable software exists), i.e., multiplying by an unitary matrix O of order r, we can convert $\hat{E}_n$ into the form

$$\hat{E}_n O \triangleq H = (\underset{\sim}{u}_{q+1}, \underset{\sim}{u}_{q+2}, \ldots, \underset{\sim}{u}_p) \tag{3.11}$$

where $\underset{\sim}{u}_{q+i} = (u_{1,q+i}, \ldots, \underset{\sim}{u}_{q+i,q+i}, 0, \ldots, 0)'$, $i = 1, \ldots, r$, with $u_{q+i,q+i} \neq 0$ (with probability 1). We may choose $\underset{\sim}{u}_{q+1}$ as an estimate of $(g_1, \ldots, g_{q+1})'$, solve the equation

$$B(z) = u_{q+1q+1} z^q + \ldots + u_{1,q+1} = 0 \tag{3.12}$$

obtain the roots in the form

$$\hat{\rho}_k e^{i\hat{\tau}_k}, \quad k = 1, \ldots, q \tag{3.13}$$

and choose $\hat{\tau}_1, \ldots, \hat{\tau}_q$ as estimates of $\tau_1, \ldots, \tau_q$.

To obtain the asymptotic distribution of the estimates, we introduce some conditions.

(A_1) The second moment of $\underset{\sim}{x}(t)$ exists.

(A_2) The fourth moment of $\underset{\sim}{x}(t)$ exists.

(A_3) $\mathrm{cov}[\mathrm{Re}\ s(t)] = \mathrm{cov}[\mathrm{Im}\ s(t)] = 2^{-1}\mathrm{Re}\ \Gamma$

$$E[(\mathrm{Re}\ s(t))(\mathrm{Im}\ s(t))'] = -E[(\mathrm{Im}\ s(t))(\mathrm{Re}\ s(t))'] = 2^{-1}\ \mathrm{Im}\ \Gamma$$

(A_4) $\mathrm{cov}[\mathrm{Re}\ \epsilon(t)] = \mathrm{cov}[\mathrm{Im}\ \epsilon(t)] = 2^{-1}\sigma^2 I_p$

$$\mathrm{cov}[\mathrm{Re}\ \epsilon(t),\ \mathrm{Im}\ \epsilon(t)] = 0$$

(A_5) $E(\mathrm{Re}\ \epsilon_k(t))^4 = E(\mathrm{Im}\ \epsilon_k(t))^4 = \frac{3}{4}\sigma^2, \quad k = 1,\dots,p$

$$E(\mathrm{Re}\ \epsilon_k(t))^2(\mathrm{Re}\ \epsilon_h(t))^2 = E(\mathrm{Im}\ \epsilon_k(t))^2(\mathrm{Im}\ \epsilon_h(t))^2 = 4^{-1}\sigma^4$$

$$k \neq h = 1,\dots,p$$

$$E(\mathrm{Re}\ \epsilon_k(t))^2(I_m\ \epsilon_h(t))^2 = 4^{-1}\sigma^4, \quad k,h = 1,\dots,p$$

and all other kinds of fourth moments are zero.

The following theorems concerning the asymptotic properties of the estimators are established in Bai, Miao and Rao (1990).

<u>Theorem 1</u>. Under the condition (A_1),

$$\hat{\underset{\sim}{\tau}} = (\hat{\tau}_1,\dots,\hat{\tau}_q)' \xrightarrow{a.s.} \underset{\sim}{\tau} = (\tau_1,\dots,\tau_q)'. \tag{3.14}$$

<u>Theorem 2</u>. Under the conditions (A_2)-(A_5) the limiting distribution of

$$\hat{\Delta}_N + j\hat{T}_n \triangleq \sqrt{N}(\hat{\rho}_1-\rho_1,\dots,\hat{\rho}_q-\rho_q) + j\sqrt{N}(\hat{\tau}_1-\tau_1,\dots,\hat{\tau}_q-\tau_q) \tag{3.15}$$

is q-variate complex normal with zero mean and covariance matrix

$$G^{-1}[\sigma^4\Gamma^{-1}(A^*A)^{-1}\Gamma^{-1} + \sigma^2\Gamma^{-1}](G^{-1})^* \tag{3.16}$$

where

$$G = \mathrm{diag}(D(\tau_1),\ldots,D(\tau_q))$$

$$D(\tau_k) = g_{q+1} \prod_{i=1,i\neq k}^{q} (e^{j\tau_k} - e^{j\tau_i})e^{j\tau_k}, \quad k = 1,\ldots,q.$$

The asymptotic covariance matrix of $\hat{T}_N = \sqrt{N}(\hat{\tau}_1-\tau_1,\ldots,\hat{\tau}_q-\tau_q)$ alone is

$$2^{-1}\mathrm{Re}[G^{-1}(\sigma^4\Gamma^{-1}(A^*A)^{-1}\Gamma^{-1}+\sigma^2\Gamma^{-1})G^{-1}] \tag{3.17}$$

Remark 1. Note that $\underset{\sim}{u}_{q+s+1},\ldots,\underset{\sim}{u}_p$ is a basis of the eigen space spanned by $\hat{\underset{\sim}{e}}_{q+s+1},\ldots,\hat{\underset{\sim}{e}}_p$, so that if the number of signals is q+s, then the corresponding g-vector with q+s+1 components is estimated by $\underset{\sim}{u}_{q+s+1}$. In practice, it is advisable to estimate the τ parameters starting with a somewhat higher value than what is indicated by the GIC criterion, study the configuration of the roots as done by Tuft and Kumaresan (1980) and take a final decision on the choice of q and estimate the τ parameters. The computational algorithm outlined above is ideally suited to such an analysis.

Remark 2. We can by postmultiplying H in (3.11) by a matrix B reduce it to the form

$$HB = (\underset{\sim}{u}_{q+1},\underset{\sim}{v}_{q+2},\ldots,\underset{\sim}{v}_p) \tag{3.18}$$

where $\underset{\sim}{v}_{q+i} = (0,\ldots,0,v_{1,q+i},\ldots,v_{q+1,q+i},0,\ldots,0)$ with i-1 zeroes in the beginning and $\|\underset{\sim}{v}_{q+i}\| = 1$. Then a possibly improved estimate of g_j in $(g_1,\ldots,g_{q+1})$ is

$$\hat{g}_j = \frac{1}{p-q}(u_{j,q+1} + v_{j,q+2} + \dots + v_{j,p}), \quad j = 1,\dots,q+1. \tag{3.19}$$

The actual improvement in the estimate (3.19) needs to be studied.

Remark 3. There is a unique vector of the type $\underset{\sim}{u}_{q+1}$ in the space generated by the columns of $\hat{E}_N$. This can be obtained in a simple way, without going through Householder transformation, as shown below. We partition $\hat{E}_N$ in the form

$$\underset{p\times(p-q)}{\hat{E}_n} = \begin{pmatrix} \underset{(q+1)\times(p-q)}{\hat{E}_{N1}} \\ \underset{(p-q-1)\times(p-q)}{\hat{E}_{N2}} \end{pmatrix} \tag{3.20}$$

and solve the equation

$$\hat{E}_{N2}\underset{\sim}{b} = 0, \quad b = (b_1,\dots,b_{p-q})'. \tag{3.21}$$

The solution of (3.21) is unique (with probability 1) apart from a scalar multiplier. Then

$$\underset{\sim}{g} = (g_1,\dots,g_{q+1})' = \hat{E}_{N1}\underset{\sim}{b}$$

apart from a scalar multiplier.

4. ESTIMATION OF DOA: DIRECT APPROACH

Let us write $\underset{\sim}{G}_i$ as defined in (3.8) in the partitioned form

$$\underset{\sim}{G}_i = (\underset{1\times(i-1)}{\underset{\sim}{0}'} : \underset{1\times(q+1)}{\underset{\sim}{g}'} : \underset{1\times(p-q-1)}{\underset{\sim}{0}'})' \tag{4.1}$$

where

$$\underset{\sim}{g} = (g_1, \ldots, g_{q+1})', \quad g_{q+1} > 0, \quad \|g\| = 1, \quad A^* g = 0. \tag{4.2}$$

Since G_i is an eigen vector of $\Sigma = A\Gamma A^* + \sigma^2 I$ corresponding to the smallest eigen value σ^2 for each i, it follows that for any chosen non-negative $\nu_1, \ldots, \nu_{p-q}$

$$\sum_1^{p-q} \nu_k \underset{\sim}{G}_k^* \Sigma \underset{\sim}{G}_k = g^*(\sum_1^{p-q} \nu_k \Sigma_{k0})g \triangleq g^* \Sigma_0 g \tag{4.3}$$

attains the minimum value $\sigma^2(\nu_1 + \ldots + \nu_{p-q})$ at $\underset{\sim}{g}$ satisfying (4.2), where Σ_{k0} is the matrix obtained from Σ by retaining only the k-th to (k+q)-th columns and rows. Then $\underset{\sim}{g}$ may be estimated by minimizing

$$\sum_1^{p-q} \nu_k \underset{\sim}{G}_k^* \hat{\Sigma} \underset{\sim}{G}_k, \text{ subject to (4.2).} \tag{4.4}$$

Writing $\hat{\Sigma}_{k0}$ for the (q+1)×(q+1) matrix obtained by retaining only the k-th to (k+q)-th columns and rows in $\hat{\Sigma}$, the expression (4.4) can be written as

$$\underset{\sim}{g}^*(\sum_1^{p-q} \nu_k \hat{\Sigma}_{k0})\underset{\sim}{g} \triangleq \underset{\sim}{g}^* \hat{\Sigma}_0 \underset{\sim}{g} \tag{4.5}$$

so that $\hat{g}$, the estimate of g, is the eigen vector of $\hat{\Sigma}_0$ corresponding to its smallest eigen value. Then we solve the polynomial equation

$$\hat{g}_{q+1} z^k + \ldots + \hat{g}_1 = 0 \tag{4.6}$$

as in Section 3. Writing the roots in the form $\hat{z}_k = \hat{\rho}_k \exp(j\hat{\tau}_k)$, $\hat{\rho}_k \geq 0$, $\hat{\tau}_k \in (0,2\pi)$, we obtain the estimate of τ_k as $\hat{\tau}_k$, $k = 1,\ldots,q$.

Note that in (4.3)

$$\Sigma_0 = \sum_1^{p-q} \nu_k \Sigma_{k0} = \sum_1^{p-q} \nu_k B^{k-1} \Gamma B^{k-1} + (\sigma^2 \sum_1^{p-q} \nu_k) I_{q+1} \tag{4.7}$$

so that even if Γ is singular, when p is large enough, the first term in (4.7) can be of rank q. The classical *MUSIC algorithm fails when* Γ is singular, but the above method works.

In order to study the limiting distribution of $\hat{\rho}_k$, $\hat{\tau}_k$, $k = 1,\ldots,q$, we make the following notations:

$$\begin{aligned}
\underset{\sim}{a}_{k0} &= (1, e^{j\tau_k}, \ldots, e^{iq\tau_k}),\ (q+1)\text{-vector.} \\
A_0 &= (\underset{\sim}{a}_{10}), \ldots, \underset{\sim}{a}_{q0}),\ (q+1)\times q \text{ matrix.} \\
B &= \mathrm{diag}(e^{j\tau_1}, \ldots, e^{j\tau_q}),\ q\times q \text{ diagonal matrix.}
\end{aligned}$$

<u>Theorem 1</u>. Under the condition (A_1)

$$\hat{\underset{\sim}{\tau}} = (\hat{\tau}_1, \ldots, \hat{\tau}_q)' \xrightarrow{\text{a.s.}} \underset{\sim}{\tau} = (\tau_1, \ldots, \tau_k)', \tag{4.8}$$

$$\hat{\underset{\sim}{\rho}} = (\hat{\rho}_1, \ldots, \hat{\rho}_q)' \xrightarrow{\text{a.s.}} \underset{\sim}{\rho} = (1, \ldots, 1)'. \tag{4.9}$$

<u>Theorem 2</u>. Under the condition (A_2), the joint distribution of $\hat{\Delta}_n + j\hat{T}_n$, where

$$\hat{\Delta}_n = \sqrt{N}(\hat{\rho}_1 - 1, \ldots, \hat{\rho}_q - 1)$$
$$\hat{T}_n = \sqrt{N}(\hat{\tau}_1 - \tau_1, \ldots, \hat{\tau}_q - \tau_q),$$

tends to that of

$$Q^{-1}\left(\sum_1^{p-q} \nu_k B^{k-1} \Gamma \bar{B}^{k-1}\right)^{-1} (A_0^* A_0)^{-1} A_0^* \left(\sum_1^{p-q} \nu_k (R_3^{(k)} + A_0 B^{k-1} R_2^{(k)})\right) \tag{4.10}$$

where $(R_2^{(k)}, R_3^{(k)})$ has a multivariate complex normal distribution and

$$Q = \mathrm{diag}\left(\sum_{k=1}^{q+1} (k-1) g_k e^{j\tau_1(k-1)}, \ldots, \sum_{k=1}^{q+1} (k-1) g_k e^{j\tau_q(k-1)}\right). \tag{4.11}$$

Let R be a $p \times p$ Gaussian random matrix with the underlying variance σ^4 and W be a $q \times p$ random matrix whose columns are iid multivariate complex normal variables with mean zero and covariance matrix $\sigma^2\Gamma$.

<u>Theorem 3</u>. Under the conditions (A_2)-(A_5), the random variables $R_2^{(k)}$ and $R_3^{(k)}$ in (4.10) can be characterized as follows:

(i) $R_2^{(k)}$ is the submatrix of W formed by its $(k+1)$-th to $(k+q+1)$-th columns.

(ii) $R_3^{(k)}$ is the submatrix formed by the (k+1)-th to (k+q+1)-th comumns and rows of R.

(iii) $(R_2^{(1)},\ldots,R_2^{(p-q)})$ and $(R_3^{(1)},\ldots,R_3^{(p-q)})$ are independently distributed.

From Theorems 2 and 3, it follows that the limiting distribution of $\hat{\Delta}_n + j\hat{T}_n$ is q-variate complex normal with mean zero and covariance matrix

$$Q^{-1}\Big(\sum_{1}^{p-q} \nu_k B^{k-1}\Gamma\, \overline{B}^{\,k-1}\Big)^{-1} V\Big(\sum_{k=1}^{p-q} \nu_k B^{k-1}\Gamma\, \overline{B}^{\,k-1}\Big)Q^{-1} \tag{4.12}$$

$$V = (A_0^*A_0)^{-1}A_0^*\sum_k \sum_s \nu_k\nu_s F_{ks} g_{ks}\sigma^4 A_0(A_0^*A_0)^{-1} + \sum_k \sum_s \nu_k\nu_s\sigma^2\Gamma\, g_{ks} \tag{4.13}$$

$$g_{ks} = \sum_{m_1=s+m_2-k} g_{m_1}\overline{g}_{m_2}, \quad k,s = 1,\ldots,p-q \tag{4.14}$$

$$F_{ks} = \begin{cases} J^{k-s} & \text{if } k \geq s \\ (J^{s-k})' & \text{if } k < s \end{cases} \tag{4.15}$$

where J is a $(p-q)\times(p-q)$ matrix with only the entries in the diagonal above the main diagonal as unities and rest as zeroes. The asymptotic covariance of $\hat{T}$ alone is 2^{-1} times the real part of (4.12)

Remark 4.1. It is seen that the expressions for the covariance matrices of the estimates $\hat{\underset{\sim}{\tau}}$ and $\hat{\underset{\sim}{\rho}}$ given in (3.16) and (4.12) are very complicated. Perhaps, here we have a case for the bootstrap method for estimating the covariance matrix. The numerical computations involved in the bootstrap method will be considered in a later communication.

<u>Remark 4.2</u>. It is seen from (1.1) that the conjugate vector $\bar{\underset{\sim}{x}}(t)$ has the model

$$\bar{\underset{\sim}{x}}(t) = \bar{A}\,\bar{\underset{\sim}{s}}(t) + \bar{\underset{\sim}{n}}(t), \quad t = 1,\dots,N. \tag{4.16}$$

with the covariance matrix of $\bar{\underset{\sim}{x}}(t)$ as

$$\bar{\Sigma} = \bar{A}\,\bar{\Gamma}\,\bar{A}^{*} + \sigma^2 I. \tag{4.17}$$

Then

$$\underset{\sim}{G}_i = (0,\dots,0,g_{q+1},\dots,g_1,0,\dots,0) \tag{4.18}$$

which is similar to $\underset{\sim}{G}_i$ of (3.8) with the components of the same $\underset{\sim}{g}$ vector written in the reverse order, is an eigen vector corresponding to the smallest eigen value of $\bar{\Sigma}$. We can estimate $g_{q+1},\dots,g_1$ using the model (4.16) in the same way as $g_1,\dots,g_{q+1}$ is estimated using the model (1.1). We have a developed method of combining the information proved by the models (1.1) and (4.16) in obtaining an efficient estimate of $g_1,\dots,g_{q+1}$, which will be given in a later communication.

REFERENCES

Anderson, T. W. (1963). Asymptotic the~ry for principle component analysis. *Ann. Math. Statist.*, 34, 122-148.

Bai, Z. D., Krishnaiah, P. R. and Zhao, L. C. (1987). On the direction of arrival estimation. Technical Report No. 87-12, Center for Multivariate Analysis, University of Pittsburgh.

Bai, Z. D., Miao, B. Q. and Rao, C. Radhakrishna (1990). Estimation of direction of arrival of signals: Asymptotic results. In *Spectrum Estimation* (ed. S. Haykin).

Bienvenu, G. (1979). Influence of the spatial coherence of the background noise on high resolution passive methods. *Proc. IEEE ICASSP*, pp. 306-309.

Ligget, W. S. (1973). Passive sonar: Fitting models to multiple time series. In *Signal Processing* (J. W. Griffith et al. eds), 327-345. Academic Press, New York.

Paulraj, A. and Kailath, T. (1986). Eigenstructure methods for direction of arrival estimation in the presence of unknown noise fields. *IEE Transaction on ASSP*-34 No. 1, pp. 13-20.

Rao, C. Radhakrishna (1973). *Linear Statistical Inference and Its Applications*, (Second Edition). Wiley, New York.

Rao, C. Radhakrishna (1983). Likelihood ratio tests for relationship between two covariance matrices. In *Studies in Econometrics, Time Series and Multivariate Statistics*, (Eds. T. Amemiya, S. Karlin and L. Goodman). Academic Press, New York, 529-544.

Rao, C. R. and Wu, Y. (1989). A strongly consistent procedure for model selection in a regression problem. *Biometrika*, 76, 369-374.

Schmidt, R. O. (1981). A signal subspace approach to multiple source location and spectral estimation. Ph.D. dissertation, Stanford University, Stanford, CA.

Tufts, D. W. and Kumaresan, R. (1980). Data-adaptive principal component signal processing. *Proc. 19th IEEE Conf. Decision Control, Albuquerque, NM*, 949-954.

Wax, M. and Kailath, T. (1984). Determination of the number of signals by information theoretic criteria. *IEEE Trans. Acoustics Speech Signal Processing*, ASSP-33, 387-392.

Wax, M., Shan, T. J. and Kailath, T. (1984). Spatio-temporal spectral analysis by eigenstructure methods. *IEEE Trans. Acoustics Speech Signal Processing*, ASSP-32, 817-827.

Zhao, L. C., Krishnaiah, P. R. and Bai, Z. D. (1986a). On detection of the number of signals in presence of white noise. *J. Multvariate Anal.*, 20, 1-25.

Zhao, L. C., Krishnaiah, P. R. and Bai, Z. D. (1986b). On detection of the number of signals when the noise covariance matrix is arbitrary. *J. Multivariate Anal.*, 20, 26-49.

Unclassified

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE

READ INSTRUCTIONS BEFORE COMPLETING FORM

1. REPORT NUMBER: ARO 26792.2-MA-SDI	2. GOVT ACCESSION NO.
3. RECIPIENT'S CATALOG NUMBER	
4. TITLE (and Subtitle): Spectral Analytic Methods for the Estimation of Number of Signals and Directions of Arrival	5. TYPE OF REPORT & PERIOD COVERED: Technical - November 1989
	6. PERFORMING ORG. REPORT NUMBER: 89-49
7. AUTHOR(s): Z. D. Bai and C. Radhakrishna Rao	8. CONTRACT OR GRANT NUMBER(s): DAAL03-89-K-0139
9. PERFORMING ORGANIZATION NAME AND ADDRESS: Center for Multivariate Analysis, 123 Pond Laboratory, Penn State Univ., University Park, PA 16802	10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS
11. CONTROLLING OFFICE NAME AND ADDRESS: U.S. Army Research Office, Research Triangle Park, NC 27709	12. REPORT DATE: November 1989
	13. NUMBER OF PAGES: 18
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office)	15. SECURITY CLASS. (of this report): Unclassified
	15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Akaike information criterion, Direction of arrival, General information criterion, Signal processing.

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

Consider the model

$$\underset{\sim}{x}(t) = A\underset{\sim}{s}(t) + \underset{\sim}{n}(t), \quad t = 1,\dots,N$$

where at time t, $\underset{\sim}{x}(t)$ is a p-complex vector of observations, $\underset{\sim}{s}(t)$ is a q-complex vector of unobservable signals emitted from q sources and $\underset{\sim}{n}(t)$ is a

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S/N 0102-014-6601

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered) noise p-complex vector variable. It is assumed that $\{\underset{\sim}{s}(t)\}$ and $\{\underset{\sim}{n}(t)\}$ are independent sequences with

$$E[\underset{\sim}{n}(t)\underset{\sim}{n}(t)^*] = \sigma^2 I, \quad E[\underset{\sim}{s}(t)\underset{\sim}{s}(t^*)] = \Gamma$$
$$\Rightarrow E[\underset{\sim}{x}(t)\underset{\sim}{x}(t)^*] = A\Gamma A^* + \sigma^2 I$$

The matrix A has a special structure with its k-th column $\underset{\sim}{a}_k$ of the form

$$\underset{\sim}{a}_k' = (1, e^{-j\omega_0\tau_k}, \ldots, e^{-j\omega_0(p-1)\tau_k})$$

where $j = \sqrt{-1}$, $\tau_k = c^{-1}\Delta\sin\theta_k$, θ_k being the direction of arrival of the signal from the k-th source. The problem is the estimation of q (the number of sources) and θ_i's (the directions of arrival).

The paper presents an information theoretic criterion to decide on q, the number of sources, and a spectral analytic method to estimate the τ_i's. The proposed method is straight forward and has some advantages over the search methods proposed in the literature for the latter problem. For instance, our method works even when the signals are coherent, i.e., the covariance matrix Γ of signal is singular, whereas the MUSIC algorithm generally used in searching for the estimates of τ_k is not applicable.